MODELING OF ROTOR DYNAMIC SYSEMS WITH SPATIAL SHAFTS LOCATION

Mikhail LEONTIEV
PhD, professor of MAI (Moscow)
Alexander IVANOV
candidate of science, associate professor of MAI (Moscow)
SERGEY DEGTIAREV
Course leader of Alfa Transit Co., Ltd. (Khimki)

Abstract
Rod models of rotor systems and modal methods of their analysis are still in demand when solving of real-world problems of rotor dynamics of rotating machines. There are some algorithms and programs of such models analysis. Mainly the tasks at axisymmetric statement are solved. At the same time there are a lot of constructions where multiple shaft models of rotor systems with spatial axis position of their subsystems – shafts, case elements and links between them - should be built. The article considers creation of motion equations of such systems that allow calculations of joint bending-longitudinal-torsional vibrations of any spatial system including subsystems with parallel, skew, crossed axes. An example of such system calculation in the program system Dynamics R4 for calculation of dynamic characteristics of rotating machines is given.
Key words: rotor dynamics, spatial rod systems, stiffness and inertia matrixes, DYNAMICS R4

Introduction

Modern tendencies in solving of rotor dynamic tasks of gas turbine engines determine necessity of calculations of joint bending-longitudinal-torsional vibrations for systems with spatial location of rotation axes. Such systems may be found in jet engines, helicopters transmissions, wind setting transmissions, etc.
Earlier versions of specialized programs for solution of practical tasks of rotor dynamics were used mainly to calculate rod systems at axisymmetric statement. Such statement is characterized by coaxial location of rotor and stator axes, axisymmetric suspension, location of inertia elements, etc. Work in this direction has been hold by D. Chronin [1], A.Ivanov, M.Leontiev [2in MAI, V.Bauyer [3] in CIAM and other researchers.

Appearance of high-performance of finite-element program systems allowed solving of tasks taking into account breakdown of axial symmetry of rotor systems. However, their use up to nowadays has been connected with high laboriousness of modeling and limitations for analysis of nonlinear tasks at unsteady statement with such specific elements of rotor systems as journal bearings, elastic-damping supports, clearances, rolling bearings, etc. So algorithms and programs for modeling and analysis of rod rotor systems continue to be used and developed. Such systems are described by special elements – beams, shells, different types of bearings and so on.

Coupled vibration calculation of rotor systems with crossed axes has high priority at modeling of systems including gear couplings. In multiplying gears, reduction gears there may be quite a lot of shafts (subsystems), spaced and connected by gear pairs. Axes may be parallel (cylindrical gear couplings), crossed (bevel gear couplings), skew (hypoid gear couplings).

The article presents mathematical models and algorithms of spatial rod systems calculation. An example of their use in the Dynamics R4 program system [4] is given.

Motion equation

General dynamic equation for discrete linear oscillation systems in matrix form is usually written in this way:

$$M \cdot \ddot{q} + C \cdot \dot{q} + K \cdot q = Q(t) \ , \qquad (1)$$

where M – inertia matrix, C – damping and gyroscopic matrix, K – stiffness matrix, Q – external forces matrix, q – displacements matrix.

The task about eigenvalues and vectors without damping is described by the equation:

$$M \cdot \ddot{q} + K \cdot q = 0 \,. \qquad (2)$$

Equation (2) may be also written through flexibility matrix.

$$A \cdot M \cdot \ddot{q} + I \cdot q = 0 \,, \qquad (3)$$

where A – flexibility matrix, I – identity matrix.

Equations (2) and (3) are easily solved by the existing mathematical programs directly. We consider the case when dynamic system motion is described by equation (2).

Let us examine mathematical models and algorithms of such matrixes creation for spatial systems with crossed axes of subsystems.

Definition of coordinate system

Let generalized model consist of n subsystems connected between themselves and with stationary foundation by m links. Subsystems include structural elements of construction, described by beam, shell and inertia final elements. Rotors, cases, foundations may be subsystems. It is supposed that axial lines of subsystems are crossed lines.

Fig. 1 shows relative position of coordinate systems connected with the subsystems origin. The global (main) right Cartesian rectangular coordinate system *OXYZ*, and with subsystems – analogous local systems $(O'X'Y'Z')^{(i)}$ $i=1\ldots n$.

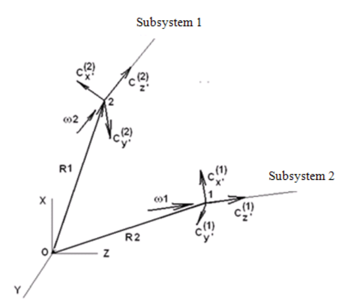


Fig. 1

X (X') and Y(Y') axes in all subsystems are transverse axes, and Z(Z') axis – a longitudinal one. Position of every local coordinate system relative to the main one is set by **Ri** vector and three unit vectors $C_{x'}^{(i)}$, $C_{y'}^{(i)}$, $C_{z'}^{(i)}$.
Vector **Ri** determines position of initial *O*' point of the local coordinate system and set by the projections column *R* along the axis of main coordinate system:

$$R = (Rx, Ry, Rz)^T \qquad (4)$$

Superscript T in R record means transposition operation in the sequel.

Unit vectors *Cx', Cy', Cz'* determine directions of corresponding local coordinate axes *X', Y', Z'* and are set by the corresponding projections columns (direction cosines):

$$Cx'=(Cx'x,\ Cx'y,\ Cx'z)^T;$$
$$Cy'=(Cy'x,\ Cy'y,\ Cy'z)^T; \qquad (5)$$
$$Cz'=(Cz'x,\ Cz'y,\ Cz'z)^T.$$

Columns Cx', Cy', Cz' form the matrix of direction cosines C:

$$C=\begin{pmatrix}Cx' & Cy' & Cz'\end{pmatrix}=\begin{pmatrix}Cx'x & Cy'x & Cz'x\\ Cx'y & Cy'y & Cz'y\\ Cx'z & Cy'z & Cz'z\end{pmatrix}. \qquad (6)$$

Using matrix C (6), conversions of local projections into global projections and vice-versa for any vector **V** are written in the following way:

$$V = C\cdot V';\quad V' = C^T\cdot V. \qquad (7)$$

Axes orientation of local coordinate system may be set by three Euler angles ψ, θ and φ, where ψ - precession angle (angle of initial turn of X and Y axes around Z axis); θ - nutation angle (deviation angle of Z' axis from Z axis, obtained by rotation along new position of Y axis); φ – angle of proper rotation (rotation angle about Z' axis of the coordinate system obtained as a result of previous operations).

Matrix C of the direction cosines of the local coordinate system axes may be written through trigonometric functions of Euler angles (8):

$$C=\begin{pmatrix}\cos\varphi\cdot\cos\theta\cdot\cos\psi-\sin\varphi\cdot\sin\psi & \cos\varphi\cdot\cos\theta\cdot\sin\psi+\sin\varphi\cdot\cos\psi & -\cos\varphi\cdot\sin\theta\\ -\sin\varphi\cdot\cos\theta\cdot\cos\psi-\cos\varphi\cdot\sin\psi & -\sin\varphi\cdot\cos\theta\cdot\sin\psi+\cos\varphi\cdot\cos\psi & \sin\varphi\cdot\sin\theta\\ \sin\theta\cdot\cos\psi & \sin\theta\cdot\sin\psi & \cos\theta\end{pmatrix}^T$$

(8)

If orientation of the local system axes is set by consecutive rotation about X axis, intermediate position of Y' axis and final position of Z' axis on α, β and γ angles correspondingly, so transposed matrix C may be obtained by multiplication of three rotation matrixes (9):

$$C^T=\begin{pmatrix}\cos\gamma & \sin\gamma & 0\\ -\sin\gamma & \cos\gamma & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}\cos\beta & 0 & -\sin\beta\\ 0 & 1 & 0\\ \sin\beta & 0 & \cos\beta\end{pmatrix}\cdot\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\alpha & \sin\alpha\\ 0 & -\sin\alpha & \cos\alpha\end{pmatrix}. \qquad (9)$$

Subsystems rotation is set by the column of angular speeds $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$, where ω_i - angular rotating speed of i subsystem with '+' or '-' sign. In the accepted right coordinate system the positive rotation is clockwise, if we look at the direction of unit vector ***Cz'*** of longitudinal axis of the subsystem. Fig. 1 shows position of the rotating speed vector of the subsystem for positive rotation. Changing the direction (sign) of rotation, direction of vector **ω** also changes.

Description of spatial rod system

Generalized model of the rod system consists of several spatial subsystems connected between themselves and stationary foundation by elastic links. Every subsystem is presented as elementary parts in tandem separated by sections. Any section is characterized by set of two indexes (s,i), where s - subsystem number, i – section number in subsystem, $s = 1,2\ldots nS$, $i=0,1\ldots nel(s)$, nS – subsystems number, $nel(s)$ – parts number in subsystem s. Parts numbers in the subsystem coincide with the sections numbers placed at the end of the part.

Elementary parts of the subsystems are set by the stiffness matrixes $k_i^{(s)}$, and inertia matrixes $M1_i^{(s)}$ and $M2_i^{(s)}$ of initial and final sections correspondingly. Stiffness matrixes are included into matrix equations of parts

$$q_i^{(s)}\cdot k_i^{(s)}=Qr_i^{(s)}, \qquad (10)$$

Inertia matrixes $M1_i^{(s)}$ and $M2_i^{(s)}$ are included into matrix equations describing inertial loads of elementary parts applied to the first and final sections.

$$Qi_{i-1}^{(s)}=-M1_i^{(s)}\cdot\ddot{q}_{i-1}^{(s)};$$
$$Qi_i^{(s)}=-M2_i^{(s)}\cdot\ddot{q}_i^{(s)}, \qquad (11)$$

where $Qi_{i-1}^{(s)}$, $\ddot{q}_{i-1}^{(s)}$, $Qi_i^{(s)}$, $\ddot{q}_i^{(s)}$ - columns of inertial loads and displacements accelerations of the first and final sections of the elementary part under consideration.

Links are set by stiffness matrixes $k^{(L)}$, which are included into matrix equations describing links deformations

$$Qr_2^{(L)}=-k^{(L)}\cdot q_2^{(L)},\ L = 1,2\ldots nL, \qquad (12)$$

where nL – links number; $k^{(L)}$ - stiffness matrix of the link with an order number L; $q_2^{(L)}$ - displacements column of the final section; $Qr_2^{(L)}$-column of link reactions in final section appearing at displacements of the final section set by the column $q_2^{(L)}$, under condition of restraint of the first section.

The first and final section of the link coincide with subsystem sections set by pairs of indexes ($s1$, $i1$) and ($s2$, $i2$) correspondingly.

Relative position of subsystems sections, stiffness and inertia matrixes of elementary parts, stiffness matrixes of links are set in local coordinate systems, oriented in certain way, relatively the global coordinate system OXYZ that is common for the whole system. Vectors and matrixes conversion because of the move from local coordinate systems into the global coordinate system is described above.

After move into the global coordinate system, position of sections of s-th subsystem is set by radius-vectors $R_i^{(s)}$, as shown in Fig. 2.

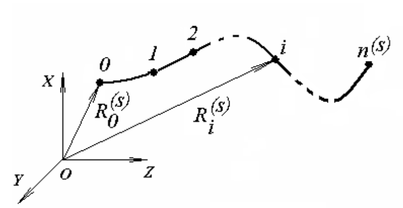


Fig. 2

Sections displacements and inner loads acting on the section from the part following the section, are presented as columns

$$q=\begin{pmatrix}u\\ \varphi\end{pmatrix};\qquad Q=\begin{pmatrix}P\\ T\end{pmatrix}, \qquad (13)$$

where u,φ,P,T - columns of projections vectors of linear displacements, angular displacement, force and moment on coordinate axes:

$$u=(u_x,u_y,u_z)^T,\ \varphi=(\varphi_x,\varphi_y,\varphi_z)^T,$$
$$P=(P_x,P_y,P_z)^T,\ T=(T_x,T_y,T_z)^T.$$

Links reactions Qr, inertia loads Qi and external loads ΔQ are given similarly:

$$Qr = \begin{pmatrix} Pr \\ Tr \end{pmatrix}; \; Qi = \begin{pmatrix} Pi \\ Ti \end{pmatrix}; \; \Delta Q = \begin{pmatrix} \Delta P \\ \Delta T \end{pmatrix}. \qquad (14)$$

Along with the shown above notations, the matrixes O_n, I_n, O, I, $W(r)$ will be used in the following descriptions. They will be determined in this way:
O_n and I_n – square zero and unit n-th order matrixes;
O and I – square zero and unit matrixes whose orders are determined from the context;
$W(r)$ – antisymmetric square matrix used at matrix form of vector products. Matrix elements $W(r)$ are determined by projections of vector r on coordinate axes

$$W(r) = \begin{pmatrix} 0 & r_Z & -r_X \\ -r_Z & 0 & r_Y \\ r_X & -r_Y & 0 \end{pmatrix}. \qquad (15)$$

Taking into account accepted notations of coordinates, inertia matrixes of the first and final sections of *i-th* part of subsystem *s* may be written as the following

$$M1_i^{(s)} = \begin{pmatrix} m1 \cdot I_3 & W(S1) \\ W(S1)^T & J1 \end{pmatrix}_i^{(s)}, \quad S1 = \begin{pmatrix} S1_x \\ S1_y \\ S1_z \end{pmatrix}_1^{(s)};$$

$$M2_i^{(s)} = \begin{pmatrix} m2 \cdot I_3 & W(S2) \\ W(S2)^T & J2 \end{pmatrix}_i^{(s)}, \quad S2 = \begin{pmatrix} S2_x \\ S2_y \\ S2_z \end{pmatrix}_1^{(s)},$$

where $m1_i^{(s)}, m2_i^{(s)}$ - masses attached to the first and final section of the elementary part;
- columns of vectors projections of static moments of masses $m1_i^{(s)}$ and $m2_i^{(s)}$ relatively to the first and final section of the elementary part correspondingly;

$$J1_i^{(s)} = \begin{pmatrix} J1_{xx} & J1_{xy} & J1_{xz} \\ J1_{yx} & J1_{yy} & J1_{yz} \\ J1_{zx} & J1_{zy} & J1_{zz} \end{pmatrix}_i^{(s)},$$

$$J2_i^{(s)} = \begin{pmatrix} J2_{xx} & J2_{xy} & J2_{xz} \\ J2_{yx} & J2_{yy} & J2_{yz} \\ J2_{zx} & J2_{zy} & J2_{zz} \end{pmatrix}_i^{(s)}$$ - matrixes of inertia moments of masses $m1_i^{(s)}$ and $m2_i^{(s)}$ relatively to the first and final section of the elementary part correspondingly.

Matrix of static stiffness of elementary part

Stiffness of *i-th* elementary part of the subsystem *s* is described by the full matrix $K_i^{(s)}$ that is included into matrix equation

$$\begin{pmatrix} Q_{i-1}^{(s)} \\ Q_i^{(s)} \end{pmatrix} = -K_i^{(s)} \cdot \begin{pmatrix} q_{i-1}^{(s)} \\ q_i^{(s)} \end{pmatrix} \qquad (16)$$

Matrix $K_i^{(s)}$ may be presented as blocks

$$K_i^{(s)} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}_i^{(s)}. \qquad (17)$$

Matrixes of static stiffness of elastic link between subsystems

The whole stiffness $K^{(L)}$ of the elastic link with the order number L is determined by matrix equation

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}^{(L)} = -K^{(L)} \cdot \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}^{(L)}, \qquad (18)$$

where $q_1^{(L)} = (q_{i1}^{(s1)})^{(L)}$ и $q_2^{(L)} = (q_{i2}^{(s2)})^{(L)}$ –columns of displacements of the first and final link section correspondingly;
$Q_1^{(L)} = (Q_{i1}^{(s1)})^{(L)}$ и $Q_2^{(L)} = (Q_{i2}^{(s2)})^{(L)}$ - columns of link reactions in the indicated sections.
Matrix $K^{(L)}$ may be presented as blocks

$$K^{(L)} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}^{(L)}. \qquad (19)$$

Blocks included into the expression (19) are calculated through the known stiffness matrix $K^{(L)}$ of the free final link section at the fixed first section using the following equations:

$$K_{11}^{(L)} = D^T \cdot K_{22}^{(L)} \cdot D; \qquad K_{12}^{(L)} = -D^T \cdot K_{22}^{(L)};$$
$$K_{21}^{(L)} = (K_{12}^{(L)})^T; \quad K_{22}^{(L)} = k^{(L)}, \qquad (20)$$

where

$$D = \begin{pmatrix} I_3 & W(r) \\ 0_3 & I_3 \end{pmatrix}, \quad r = (R_{i2}^{(s2)})^{(L)} - (R_{i1}^{(s1)})^{(L)};$$

$(R_{i2}^{(s2)})^{(L)}$ and $(R_{i1}^{(s1)})^{(L)}$ - radii-vectors of the first and final link section.

Creation of stiffness matrix of dynamic system

Full stiffness matrix of the whole system is obtained by superposition of stiffness matrixes of the elementary parts and links by common way used in all finite-element systems.

Matrix of the subsystem *s* is obtained by superposition of matrixes of its elementary parts using the following method:

$$K^{(s)} = \begin{pmatrix} K_{11} & K_{12} & O_6 & \cdots & O_6 \\ K_{21} & K_{22} & K_{23} & \cdots & O_6 \\ O_6 & K_{32} & K_{33} & \cdots & O_6 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_6 & O_6 & O_6 & \cdots & K_{n+1,n+1} \end{pmatrix}^{(s)}, \qquad (21)$$

where

$$K_{11}^{(s)} = (K_{11})_1^{(s)}; \quad K_{12}^{(s)} = (K_{12})_1^{(s)};$$
$$K_{21}^{(s)} = (K_{21})_1^{(s)}; \quad K_{22}^{(s)} = (K_{22})_1^{(s)} + (K_{11})_2^{(s)}; \quad K_{23}^{(s)} = (K_{12})_2^{(s)};$$
$$K_{32}^{(s)} = (K_{21})_2^{(s)}; \quad K_{33}^{(s)} = (K_{22})_2^{(s)} + (K_{11})_3^{(s)}; \quad \cdots$$
$$\cdots \quad K_{n+1,n+1}^{(s)} = (K_{22})_{n(s)}^{(s)}.$$

The process of the matrix creation is shown in Fig. 3. Full stiffness matrixes of the elementary parts having the size 12x12 are placed bias. Superposition places, where diagonal blocks of the full matrixes of the elementary parts are summarized, shown by darker colour and correspond to the stiffness matrixes of the subsystem inner sections dividing neighboring elementary parts. These matrixes have the size 6*6.

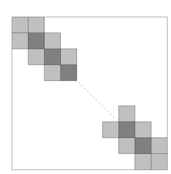


Fig. 3

The full stiffness matrix of the system consisting of n unlinked subsystems is block-diagonal matrix built from the matrixes of the subsystems included into it

$$K = diag(K^{(1)},\ K^{(2)},\dots,K^{(n)})\,. \qquad (22)$$

Fig.4 shows graphical image of such matrix.

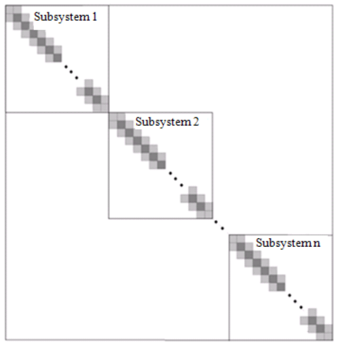


Fig. 4

Adding of the elastic link brings to summation of the blocks of the whole matrix of the added link with the blocks of the full system matrix. At addition of the link with the order number L , whose first section is section $i1$ of the subsystem $s1$, and final section is section $i2$ of the subsystem $s2$, blocks are summarized in the following way:

$$K^{(s1,s1)}_{i1,i1} \Leftarrow K^{(s1,s1)}_{i1,i1} + K^{(L)}_{11}; \quad K^{(s1,s2)}_{i1,i2} \Leftarrow K^{(s1,s2)}_{i1,i2} + K^{(L)}_{12};$$
$$K^{(s2,s1)}_{i2,i1} \Leftarrow K^{(s2,s1)}_{i2,i1} + K^{(L)}_{21}; \quad K^{(s2,s2)}_{i2,i2} \Leftarrow K^{(s2,s2)}_{i2,i2} + K^{(L)}_{22}, \qquad (23)$$

where superscripts are indexes of the block-diagonal matrix formed by the subsystems matrixes and subscripts are the indexes of sections singling out subunits of the corresponding sections inside the blocks of this matrix.

As an example, Fig. 5 gives graphical image of conversion of the system whole matrix, shown in Fig.4, at addition of the link connecting second section of the first subsystem with the fifth section of the second subsystem.

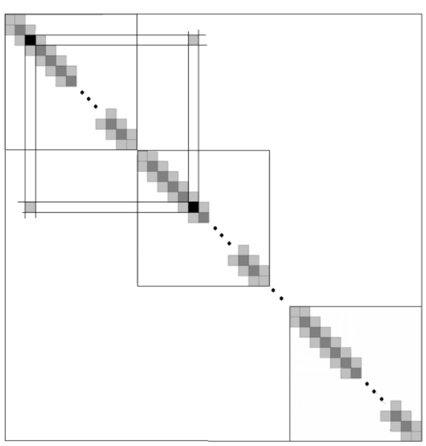


Fig. 5

As a result of the described steps, the full matrix of the system static stiffness may be obtained. Using it, external loads ΔQ may be obtained. They should be applied to the systems sections in order to obtain given displacements q of these sections. The corresponding matrix equation is

$$\Delta Q = K \cdot q\,, \qquad (24)$$

where q – column of displacements of the all system sections from zero section of the first subsystem to the final section of the last subsystem;
ΔQ - column of external loads corresponding to the displacements of the column q.

Creation of inertia matrix of dynamic system

Full system inertia M is block-diagonal matrix consisting of subsystems inertia matrixes

$$M = diag(M^{(1)}, M^{(2)}, \dots, M^{(nS)})\,. \qquad (25)$$

Subsystems inertia matrixes are also block-diagonal matrixes consisting of inertia matrixes of the sections. Inertia matrixes of the subsystem s sections are composed from inertia matrixes of the elementary parts in the following way:

$$M^{(s)}_{11} = M1^{(s)}_{1};$$
$$M^{(s)}_{22} = M2^{(s)}_{1} + M1^{(s)}_{2}; \quad M^{(s)}_{33} = M2^{(s)}_{2} + M1^{(s)}_{3};\dots, \quad M^{(S)}_{n(s),n(s)} = M2^{(s)}_{n(s)} + M1^{(s)}_{n(s)+1};$$
$$M^{(s)}_{n(s)+1,n(s)+1} = M2^{(s)}_{n(s)}. \qquad (26)$$

As a result of the described steps, the full inertia matrix of the system is obtained. Using it, inertial external loads Qi may be obtained. They should be applied to the system sections moving with the given accelerations $\ddot{q}$ in order to provide balance of the forces and moments applied to the system and, according to Dalamber principle, to go from dynamic task to the static one. The corresponding matrix equation is

$$Qi = -M \cdot \ddot{q}\,. \qquad (27)$$

Example of rod spatial system calculation

Fig. 6 shows the dynamic system consisting of three rod subsystems connected by rigid links.

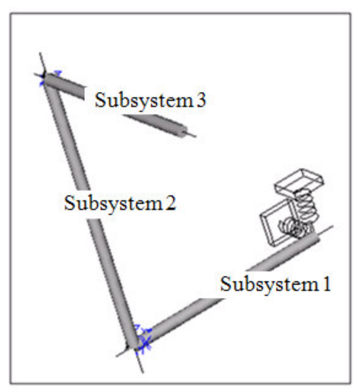


Fig. 6

Subsystems 1, 2 and 3 are normally disposed as it shown in Fig. 6. They are described by cylindrical beams with the following parameters:

- beams diameter $D_1=D_2=D_3=$ 50 mm;
- beams length $L_1=L_3=$900 мм, $L_2=$ 1200 mm;
- material density ρ=7850 kg/m^3;
- modulus of elasticity E=2.1·10^{11} N/m^2;
- Poisson's ratio μ=0.3.

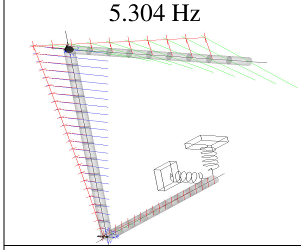
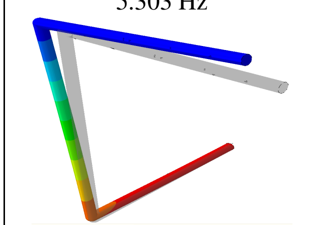
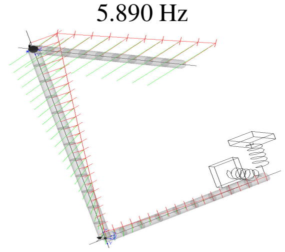
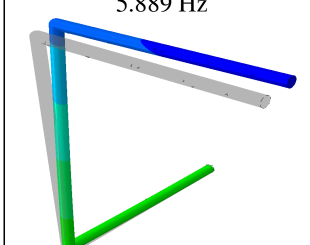
Table 1

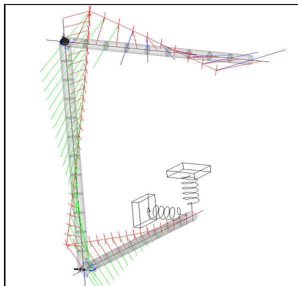
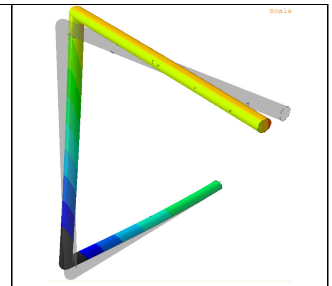
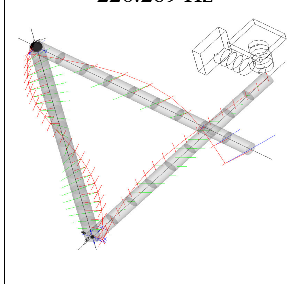
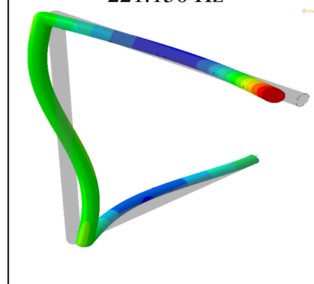
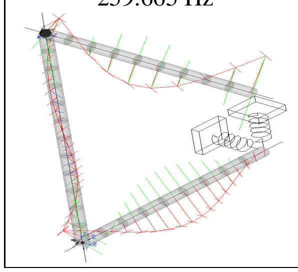
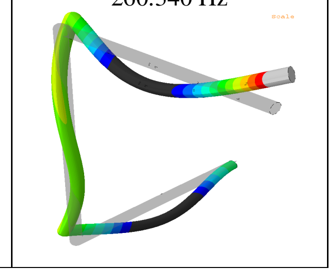
Shape number	DYNAMICS R4	FEM Beam elements	Error
	Hz	Hz	%
1	5.304	5.303	0.02
2	5.890	5.889	0.02
3	16.952	16.941	0.07
4	18.498	18.492	0.03
5	31.504	31.490	0.04
6	45.970	45.942	0.06
7	103.709	103.780	-0.07
8	125.055	125.140	-0.07
9	220.269	221.150	-0.40
10	220.917	221.570	-0.30
11	257.174	257.900	-0.28
12	259.665	260.340	-0.26

Stiffness matrixes of links between subsystems are diagonal. Stiffness coefficients of links at all freedom degrees have values of $1 \cdot 10^{11}$ N/m, that approximate them to absolutely rigid links.

Table 1 shows natural frequencies of the investigated spatial system, obtained according to the developed algorithms and in a finite-element program. The first 12 mode shapes are calculated. Only cross motions are highlighted in the Table 2. Torsional and axial displacements are not shown to avoid overloading of the pictures.

Table 2

Dynamics R4	FEM
5.304 Hz	5.303 Hz
	
5.890 Hz	5.889 Hz
	
16.952 Hz	16.941 Hz

	
220.269 Hz	221.150 Hz
	
259.665 Hz	260.340 Hz
	

Comparison of the obtained results and mode shapes shows virtually full convergence of results.

Conclusions

Mathematical models, algorithms and a program modulus were developed in the Dynamics R4 program system for modeling and analysis of complicated spatial systems consisting of rod subsystems with elastic links. Application of these developments allows calculating of joint bending -longitudinal-torsional vibrations of multiple shaft dynamic systems taking into account spatial location of their shafts, cases, inertial elements, suspension, etc.

Literature

1. Д.В. Хронин. Теория и расчет колебаний в двигателях летательных аппаратов, М.: Машиностроение, 1980 г. -411 с.

2. А.В. Иванов, М.К. Леонтьев. Модальный анализ динамических систем роторов “Известия высших учебных заведений. Авиационная техника”. 2005, №3. С.31-35.

3. В.О. Бауэр, И.А. Биргер и др. Динамика авиационных двигателей, М.: Машиностроение, 1981 г. - 232 с.

4. Леонтьев М.К., Дегтярев С.А. и др. Программная система расчета динамики роторов Dynamics 4. Свидетельство об отраслевой разработке №6691. Отраслевой фонд алгоритмов и программ. Государственный информационный центр информационных технологий. Министерство образования Российской Федерации. 2006